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Energy-regenerative model predictive control

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Abstract

This paper presents some solution approaches to the problem of optimal energy-regenerative model predictive control for linear systems subject to stability and/or dissipativity constraints, as well as hard constraints on the state and control vectors. The problem is generally non-convex in the objective and some of the constraints, thereby resulting in a non-convex optimization problem to be solved at each time step. Multiple extended convex *relaxation* approaches are considered. As a result, a more conservative semi-definite programming problem is proposed to be solved at each time step. The feasibility and stability of the resulting closed-loop system are also examined. The approaches are validated using a numerical example of maximizing energy regeneration from a single degree of freedom vibrating system subject to a level-set constraint on some performance metric characterizing the quality of vibration isolation achieved by the system. The constraint is described in terms of an upper bound on the $\mathcal{L}_2$−gain of the system from the input to a vector of appropriately selected system outputs.
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1. Introduction

The development of energy-regenerative engineering systems is on the rise due to the ever-increasing awareness of limited resources and the need to recuperate energy that would otherwise be wasted system operation. Many human activities involve converting energy from one domain to another. For example, the conversion of mechanical energy to electrical energy, which can then power computers, light, motors, etc. The input energy propels the work and is mostly converted to heat or follows the product in the process as output energy. Energy recovery systems harvest the output power and provide it as input power to the same or another process [1]. Examples of such

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systems include using heated water from sources like steel mills as heating for homes, regenerative braking, energy harvesting from vibrating systems, heat regenerative engines, etc.

The approaches used in the control systems of energy-regenerative systems can be seen from two perspectives: direct and indirect methods. Indirect methods utilize control systems for some primary objective and as a result extracts energy from the system. An example is a vibrating system. Any controller results in the closed-loop system being dissipative which makes the energy available for regeneration [2–4]. On the other hand, direct methods seek to directly extract energy from the system while satisfying some performance constraint. Examples of direct methods include a sliding mode control with an appropriate choice of the sliding surface [5] and model predictive control [6].

Model predictive control (MPC) refers to a class of control systems in which the current control action is obtained at each sampling instant by solving a finite (or infinite) horizon open-loop optimal control problem. While the result of the optimization is a sequence of control actions over the prediction horizon, only the first control action is applied at the current time.[1] The process is repeated at each sampling time to obtain the desired control input. Using this framework, it is easy to cope with hard constraints on controls and states. As a result, MPC has received a lot of attention in the literature for both discrete and continuous time systems [8–17].

Consequently, MPC-based solutions for energy regeneration problems are receiving a lot of research attention. Interested readers are directed to the reference [18] and the references therein for a survey of prior works in this area. Moreover, it was reported in [19] that it is troublesome to ensure stability if the problem is nonconvex, and in addition, the explicit methods are not suitable for larger problems due to extremely large state-space models. In this paper, the stability issue is tackled by explicitly imposing stability/dissipativity constraints which are then *convexified* by introducing some extended convex relaxation approaches. More concretely, this paper considers the problem of optimal energy-regenerative MPC for linear systems subject to stability/dissipativity constraints, as well as hard constraints on the state and control vectors. The problem is generally non-convex in the objective and some of the constraints, thereby resulting in a non-convex optimization problem to be solved at each time step. Some convex relaxation approaches are considered. As a result, a more conservative semi-definite programming problem is proposed to be solved at each time step. The feasibility and stability of the resulting closed-loop system are also examined. The approaches are validated using a numerical example of maximizing the power regenerated from a single degree of freedom vibrating system subject to a level-set constraint on some weighted performance metric. The constraint is described in terms of an upper bound on the $\mathcal{L}_2$-gain of the system from the input to a vector of appropriately selected system outputs.

The rest of the paper is organized as follows: notations used throughout the paper are introduced in Section 2. The problem formulation is given in Section 3. The convex relaxation procedures are described in Section 4 with the feasibility and stability of the resulting relaxed MPC problem. In Section 5, the relaxed MPC problem is extended to the output feedback case. A numerical simulation example is given in Section 6. Conclusions follow in Section 7.

2. Notations

Throughout the paper, the following notations are used: $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and positive real numbers respectively. The set of all symmetric positive definite and positive semi-definite matrices are denoted by $\mathbb{S}_{++}$ and $\mathbb{S}_+$ respectively. The Euclidean norm of

[1]Except otherwise required in some special circumstances (for example, see reference [7] and references therein).

a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted by $\|\mathbf{x}\| \triangleq \left(\mathbf{x}^T\mathbf{x}\right)^{1/2}$. The quadratic form $\|\mathbf{x}\|_P^2 \triangleq \mathbf{x}^T P\mathbf{x}$ is defined for any matrix $P \in \mathbb{S}_+$. The expression $P \preceq Q$ means that the matrix $Q - P \in \mathbb{S}_+$. The Euclidean balls $\mathbb{B}_r(0)$ and $\mathbb{B}_r(\mathbf{x}_0)$ are defined respectively for some $r \in \mathbb{R}_+$ as $\mathbb{B}_r(0) \triangleq \{\mathbf{x} : \|\mathbf{x}\| \le r\}$ and $\mathbb{B}_r(\mathbf{x}_0) \triangleq \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| \le r\}$.

3. Problem formulation

Using the MPC framework, the problem considered is to solve the optimization problem defined below at each time step:

$$\text{minimize} \quad J = \sum_{k=0}^{N} \mathbf{x}_k^T C^T \mathbf{u}_k \tag{1}$$

$$\text{subject to :} \quad \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \quad \mathbf{x}_0 = \mathbf{x}(t), \tag{2}$$

$$\|\mathbf{u}_k\| \le \overline{u}, \tag{3}$$

$$\text{stability/dissipativity constraint}, \tag{4}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m_1}$ are the system state and input matrices respectively, $\mathbf{x}_k \triangleq \mathbf{x}(t + T_s k|t) \in \mathbb{R}^n$, $\mathbf{u}_k \triangleq \mathbf{u}(t + T_s k|t)$, $k = 0, 1, \ldots N$ are the predicted future states and control sequences respectively, $\overline{u} > 0$ is a given saturation requirement on the control, and N defines the prediction horizon.

The objective function in Eq. (1) is indefinite. As a result, the value function of the above optimization cannot be used as a Lyapunov function for the closed loop system, as is the case with traditional model predictive control design. In order to ensure the stability of the ensuing closed loop system, an explicit constraint is enforced in Eq. (4) either as a pure stability constraint in the Lyapunov sense or as a "harder" dissipativity constraint. The stability constraint in Eq. (4) is expressed as a Lyapunov stability criterion as follows; there exists $V : \mathbb{R}^n \to \mathbb{R}_+, V(\mathbf{0}) = 0$ such that

$$V(\mathbf{x}((i+1)T_s)) - V(\mathbf{x}(iT_s)) \le 0, \quad \forall i = 0, 1, \ldots . \tag{5}$$

For the sake of this paper, the Lyapunov candidate function considered is a family of quadratics $V(\mathbf{x}) = \|\mathbf{x}\|_P^2$ parametrized by $P \in \mathbb{S}_{++}$. On the other hand, the dissipativity constraint in Eq. (4) is expressed as a level set constraint on the $\mathcal{L}_2-$gain of the system from an exogenous input to a controlled output vector. To facilitate subsequent developments, the following definitions concerning the $\mathcal{L}_2-$norm of a signal and the induced $\mathcal{L}_2-$norm of a mapping over extended $\mathcal{L}_2$ space are given.

Definition (*Extended* $\mathcal{L}_2-$space). Let

$$\chi_M(iT_s) = \begin{cases} \chi(iT_s), & i <= M \\ 0, & i > M, \end{cases} \tag{6}$$

$M \in \mathbb{N}$ be a truncation of the signal $\chi(iT_s) \in \mathcal{L}_2$, the extended $\mathcal{L}_2$-space is defined as

$$\mathcal{L}_{2_{e(M)}} = \left\{\chi(iT_s) | \chi_M(iT_s) \in \mathcal{L}_2, M \in \mathbb{N}\right\}, \tag{7}$$

with the associated $\mathcal{L}_2-$norm

$$\|\chi\|_{2_{e(M)}} = \left\{\sum_{i=0}^{\infty} \|\chi_M(iT_s)\|^2\right\}^{1/2} = \left\{\sum_{i=0}^{M} \|\chi(iT_s)\|^2\right\}^{1/2}. \tag{8}$$

For the sake of clarity of exposition, the subscript $e(M)$ is dropped and we simply use $\|\chi\|_2$ to denote the $\mathcal{L}_2-$norm of a signal over the extended $\mathcal{L}_2-$space for any given $M \in \mathbb{N}$.

Definition ($\mathcal{L}_2-$gain). Consider the LTI system with an exogenous input $\mathbf{w}(iT_s) \in \mathcal{R}_2^m$, and a controlled output $\mathbf{y}_k \in \mathcal{R}^r$

$$\mathbf{G} : \begin{cases} \mathbf{x}((i+1)T_s) = A\mathbf{x}(iT_s) + B\mathbf{u}(iT_s) + B_w\mathbf{w}(iT_s) \\ \mathbf{y}(iT_s) = C_y\mathbf{x}(iT_s) + D\mathbf{u}(iT_s), \end{cases} \tag{9}$$

the $\mathcal{L}_2-$gain is defined as

$$\|\mathbf{G}\|_2 = \sup_{\|\mathbf{w}(iT_s)\|_2 \neq 0} \frac{\|\mathbf{y}(iT_s)\|_2}{\|\mathbf{w}(iT_s)\|_2}. \tag{10}$$

Remark. It is well known [20,21] that if there exists $\gamma \geq 0$ and $V : \mathbb{R}^n \to \mathbb{R}_+$, $V(\mathbf{0}) = 0$ such that

$$V(\mathbf{x}((i+1)T_s)) - V(\mathbf{x}(iT_s)) \leq \gamma^2 \|\mathbf{w}(iT_s)\|^2 - \|\mathbf{y}(iT_s)\|^2, \quad \forall i = 0, 1, \ldots, \tag{11}$$

then, the $\mathcal{L}_2-$gain of the system from $\mathbf{w}(iT_s)$ to $\mathbf{y}(iT_s)$ can be upper bounded[2] by γ.

It is interesting to note that, with $\mathbf{w}(iT_s) = \mathbf{0}$, the dissipativity condition in Eq. (11) is sufficient for the stability condition in Eq. (5). Thus, without loss of generality, only the condition in Eq. (11) will be considered for the constraint in Eq. (4). For the sake of clarity of exposition in the subsequent developments, except otherwise stated, the notation $\chi(iT_s) = \chi(i)$ will be adopted. To this end, let

$$V(\mathbf{x}) = \|\mathbf{x}\|_P^2, \quad P \in \mathbb{S}_{++}, \tag{12}$$

thus, the inequality in Eq. (11) becomes

$$\|\mathbf{x}(i+1)\|_P^2 - \|\mathbf{x}(i)\|_P^2 - \gamma^2\|\mathbf{w}(i)\|^2 + \|\mathbf{y}(i)\|^2 \leq 0, \tag{13}$$

which implies that

$$\mathbf{z}^T \begin{bmatrix} -\gamma^2\mathbf{I} + B_w^T P B_w & B_w^T P(A\mathbf{x}(i) + B\mathbf{u}(i)) \\ (A\mathbf{x}(i) + B\mathbf{u}(i))^T P B_w & \|A\mathbf{x}(i) + B\mathbf{u}(i)\|_P^2 - \|\mathbf{x}(i)\|_P^2 + \|\mathbf{y}(i)\|^2 \end{bmatrix} \mathbf{z} \leq 0, \tag{14}$$

where

$$\mathbf{z} = [\mathbf{w}(i) \;\; 1]^T. \tag{15}$$

Using the Schur Complement, the above inequality yields the following sufficient conditions:

$$-\gamma^2\mathbf{I} + B_w^T P B_w \preceq 0, \tag{16}$$

$$\|A\mathbf{x}(i) + B\mathbf{u}(i)\|_Q^2 + \|\mathbf{y}(i)\|^2 - \|\mathbf{x}(i)\|_P^2 \leq 0, \tag{17}$$

where

[2] For linear systems, this condition (with a quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$) is both necessary and sufficient and popularly known as bounded-real lemma.

$$Q = P + PB_w\left(\gamma^2\mathbf{I} - \mathbf{B}_w^T\mathbf{P}\mathbf{B}_w\right)^{-1}B_w^TP. \tag{18}$$

Thus, the MPC problem in Eqs. (1) through (4) is re-written as

$$\text{minimize} \quad J = \sum_{k=0}^{N} \mathbf{x}_k^T C^T \mathbf{u}_k \tag{19}$$

$$\text{subject to :} \quad \mathbf{x}_{k+1} - A\mathbf{x}_k - B\mathbf{u}_k \in \mathbb{B}_r(0), \quad \mathbf{x}_0 = \mathbf{x}(t), \tag{20}$$

$$\|\mathbf{u}_k\| \le \overline{u}, \tag{21}$$

$$\|A\mathbf{x}_k + B\mathbf{u}_k\|_Q^2 + \|\mathbf{y}_k\|^2 - \|\mathbf{x}_k\|_P^2 \le 0. \tag{22}$$

Since it is impossible, without any form of preview, to ascertain its future values, the exogenous input is not included in the predictive model in the above MPC but instead, the mismatch between the predicted and controlled step is restricted to live within an Euclidean ball $\mathbb{B}_r(0)$ of radius $r \ge 0$ centered around the origin. Also, by including the dissipativity constraint in Eq. (22), it is guaranteed that the controlled output signal can be upper bounded by some scalar multiple of the exogenous input according to the specified attenuation level γ. It is also noteworthy that, under the traditional MPC framework where only the first of the input sequence from the optimization procedure is applied to the plant, the dissipativity constraint only needs to be enforced for $k=0$. As a result, the constraint becomes convex—since the only unknown in Eq. (22) is $\mathbf{u}_0$ in this case. Nevertheless, in some scenarios, it might be helpful to enforce the constraint for the entire (or some part of the) prediction horizon. An example of such scenario is when there is limited processing power such that it takes more than one sampling period to solve the optimization problem [7,22]. Consequently, the dissipativity constraint will be enforced for the entire prediction horizon. The following section considers some convexification approaches for the nonconvex optimization above.

4. Convex relaxation approaches

The optimization problem described by Eqs. (19) through (22) is nonconvex due to the nonconvex objective function and the constraint in Eq. (22). Unfortunately, nonconvex optimization problems have been shown to be NP-hard[3] [23,24]. Several attempts have been made to relax nonconvex optimization problems into a more tractable convex optimization problem [25–31]. In this section, relaxation methods based on convex–concave decomposition are considered. Moreover, the constraint in Eq. (22) is replaced with a more conservative contractive constraint by requiring that the state shrinks in norm. Interested readers are directed to the references [32,33] and the references therein for a complete exposition of contractive model predictive control.

4.1. Objective function

The first convexification of the objective function is obtained by linearizing about $(\mathbf{x}_0, \mathbf{u}_{-1})$, where $\mathbf{u}_{-1}$ is the applied control from the last optimization run. It is taken as zero when the

[3] NP-hard (Non-deterministic Polynomial-time hard), in computational complexity theory, is a class of problems that are, informally, "at least as hard as the hardest problems in NP".

system is just starting from rest. Thus, the following objective function is defined:

$$J_1 = \sum_{k=0}^{N} \mathbf{x}_0^T C^T (\mathbf{u}_k - \mathbf{u}_{-1}) + (\mathbf{x}_k - \mathbf{x}_0)^T C^T \mathbf{u}_{-1} \tag{23}$$

As an alternative convexification, the objective function is first decomposed into a difference of convex functions, viz:

$$J = \frac{1}{4}\sum_{k=0}^{N} \| C\mathbf{x}_k + \mathbf{u}_k \|^2 - \| C\mathbf{x}_k - \mathbf{u}_k \|^2. \tag{24}$$

Next, $\| C\mathbf{x}_k - \mathbf{u}_k \|^2$ is linearized about $(\mathbf{x}_0, \mathbf{u}_{-1})$ resulting in the following convex objective function:

$$J_2 = \frac{1}{4}\sum_{k=0}^{N} \| C\mathbf{x}_k + \mathbf{u}_k \|^2 - 2(C\mathbf{x}_0 - \mathbf{u}_{-1})^T (C(\mathbf{x}_k - \mathbf{x}_0) - (\mathbf{u}_k - \mathbf{u}_{-1})). \tag{25}$$

This convexification works because the linearization of $-\| C\mathbf{x}_k - \mathbf{u}_k \|^2$ is a global upper bound of the concave function, making $J_2 \geq J$. Therefore, minimizing J_2 will always guarantee upper bound of the minimum of J. Next, we look at the convex relaxations for the constraint in Eq. (22). Moreover, it is important to point out that the occurrence of singular arcs[4] is not inevitable with the affine objective J_1. See Section 5.6 of [34], or Chapter 8 of [35] and references therein for a description of singular arcs in optimal control problems. An obvious example is when $\mathbf{x}_0 \in \mathrm{Null}(C), \mathbf{u}_{-1} \in \mathrm{Null}(C^T)$. When this happens, the optimization problem becomes degenerative in that the objective is zero over the entire search space. The quadratic objective J_2, however, is well conditioned for all values of the "parameters" $\mathbf{x}_0, \mathbf{u}_{-1}$ but is more conservative due to the quadratic term forcing the term $\| C\mathbf{x}_k + \mathbf{u}_k \|$ to be as small as possible. Therefore, in order to avoid singular arcs while reducing conservatism, the following objective is used:

$$J = J_1 + \mathbf{1}_\phi(\mathbf{x}_0, \mathbf{u}_{-1}) J_2, \tag{26}$$

where $\mathbf{1}_\phi(\cdot,\cdot)$ is an indicator function of the singular arc condition for the objective function J_1, and is given by

$$\mathbf{1}_\phi(\mathbf{x}_0, \mathbf{u}_{-1}) = \begin{cases} 1 & \text{if } [\mathbf{u}_{-1}^T \;\; \mathbf{x}_0^T]^T \in \mathrm{Null}\big(\mathrm{blkdiag}(C^T, C)\big) \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

4.2. Stability/dissipativity constraint

First, we verify the feasibility of the constraints in Eqs. (21) and (22) by considering the feedback law $\mathbf{u}_k = K\mathbf{x}_k$. To this effect, Eq. (22) is written as

$$\mathbf{x}_k \Big((A+BK)^T Q (A+BK) - P + \big(C_y + DK\big)^T \big(C_y + DK\big) \Big) \mathbf{x}_k \leq 0, \tag{28}$$

[4]Singular arc is used here to describe the indecision that arises due to the vanishing of the objective function over the entire search space. In general, singular arc condition refers to the situation when direct application of Pontryagin's minimum principle fails to yield a complete solution. This is usually the case when the Hamiltonian depends linearly on the control.

which implies that

$$(A+BK)^T\left(P+PB_w\left(\gamma^2\mathbf{I}-B_w^TPB_w\right)^{-1}B_w^TP\right)(A+BK)$$
$$-P+\left(C_y+DK\right)^T\left(C_y+DK\right)\preceq 0. \tag{29}$$

Using the matrix inversion lemma

$$P+PB_w\left(\gamma^2\mathbf{I}-B_w^TPB_w\right)^{-1}B_w^TP=\left(P^{-1}-\gamma^{-2}B_wB_w^T\right)^{-1}, \tag{30}$$

the matrix inequality above becomes

$$(A+BK)^T\left(P^{-1}-\gamma^{-2}B_wB_w^T\right)^{-1}(A+BK)$$
$$-P+\left(C_y+DK\right)^T\left(C_y+DK\right)\preceq 0, \tag{31}$$

which, after using the Schur complement, is equivalent to

$$\begin{bmatrix} -P+\left(C_y+DK\right)^T\left(C_y+DK\right) & (A+BK)^T \\ (A+BK)^T & -P^{-1}+\gamma^{-2}B_wB_w^T \end{bmatrix}\preceq 0. \tag{32}$$

Pre- and post-multiplying by $\begin{bmatrix} P^{-1} & 0 \\ 0 & \mathbf{I} \end{bmatrix}$ yields an equivalent LMI

$$\begin{bmatrix} -P^{-1} & (AP^{-1}+BL)^T & (C_yP^{-1}+DL)^T \\ (AP^{-1}+BL) & -P^{-1}+\gamma^{-2}B_wB_w^T & 0 \\ (C_yP^{-1}+DL) & 0 & -\mathbf{I} \end{bmatrix}\preceq 0, \tag{33}$$

where

$$L=KP^{-1}. \tag{34}$$

It is clear that the feasibility of the above LMI is sufficient for ensuring the feasibility of the dissipativity constraint in Eq. (22). Now, we consider the input constraint in Eq. (21). Define the ellipsoid $\mathcal{E}=\left\{\mathbf{x}|\mathbf{x}^TP\mathbf{x}\leq 1\right\}$. It follows from Eq. (22) that $\|\mathbf{x}_{k+1}\|_P^2\leq\|\mathbf{x}_{k+1}\|_Q^2\leq\|\mathbf{x}_k\|_P^2$. Thus $\mathbf{x}_0\in\mathcal{E}\Rightarrow\mathbf{x}_k\in\mathcal{E}\ \forall k>0$, meaning that $\mathcal{E}$ is an invariant set for the system. Consequently, Eq. (22), together with $\|\mathbf{x}_0\|_P^2\leq 1$, implies that $\|\mathbf{x}_k\|_P^2\leq 1\ \forall k>0$.

Therefore,

$$\|\mathbf{u}_k\|^2=\|K\mathbf{x}_k\|^2=\|LP\mathbf{x}_k\|^2 \tag{35}$$

$$\|\mathbf{u}_k\|^2\leq\|LP^{1/2}\|^2\|P^{1/2}\mathbf{x}_k\|^2 \tag{36}$$

$$\|\mathbf{u}_k\|^2\leq\|LP^{1/2}\|^2. \tag{37}$$

Thus, in addition to Eq. (22) and $\|\mathbf{x}_0\|_P^2\leq 1$, the inequality

$$LPL^T\leq\overline{u}^2, \tag{38}$$

implies that the inequality in Eq. (21) is satisfied with the feedback control $\mathbf{u}_k=K\mathbf{x}_k$. The above inequalities are equivalent to the LMIs

$$\begin{bmatrix} -1 & \mathbf{x}_0^T \\ \mathbf{x}_0 & -P^{-1} \end{bmatrix}\preceq 0, \quad \begin{bmatrix} -\overline{u}^2I & L \\ L^T & -P^{-1} \end{bmatrix}\preceq 0. \tag{39}$$

Therefore a sufficient condition for checking the feasibility of the constraints in Eqs. (21) and (22) is to find $P=P^T\succcurlyeq 0$ and L that satisfies the LMIs in Eqs. (33) and (39). If the LMIs are not feasible, then one of four things might be the issue; the pair (A,B) is not controllable, the attenuation level γ is too small, the control saturation bound $\overline{u}$ is too small, the constraints cannot be met by a state feedback control.

Next, the following definition introduces the notion of *extended* relaxation used in this paper.

Definition. A relaxation of the minimization problem

$$z \triangleq \min\left\{c(\mathbf{x})|\mathbf{x}\in\mathbf{X}\subseteq\mathbb{R}^n\right\}$$

is another minimization problem of the form

$$z_R \triangleq \min\left\{c_R(\mathbf{x},\mathbf{y})|\mathbf{x}\in\mathbf{X}_R\subseteq\mathbb{R}^n, \mathbf{y}\in\mathbf{Y}\subseteq\mathbb{R}^m\right\}$$

with the properties

1. $\mathbf{X}\subseteq\mathbf{X}_R$
2. $c_R(\mathbf{x},\mathbf{0})\leq c(\mathbf{x})$ for all $\mathbf{x}\in\mathbf{X}$

In light of the definition above, and using the property of the convexity of the norm function $\|\cdot\|_P^2$ for all $\mathbf{x}_0\in\mathbb{R}^n$

$$\begin{aligned}\|\mathbf{x}_k\|_P^2 &\geq \|\mathbf{x}_0\|_P^2 + 2\mathbf{x}_0^TP(\mathbf{x}_k-\mathbf{x}_0)\\ &= -\|\mathbf{x}_0\|_P^2 + 2\mathbf{x}_0^TP\mathbf{x}_k,\end{aligned} \tag{40}$$

the following convex relaxation of the constraint in Eq. (22) are considered:

$$\mathbf{C}_1 : \|A\mathbf{x}_k + B\mathbf{u}_k\|_Q^2 + \|\mathbf{y}_k\|^2 + \|\mathbf{x}_0\|_P^2 - 2\mathbf{x}_0^TP\mathbf{x}_k \leq \beta_k, \tag{41}$$

$$\mathbf{C}_2 : \mathbf{x}_k^TM\mathbf{x}_k + \|\mathbf{x}_0\|_P^2 - 2\mathbf{x}_0^TP\mathbf{x}_k \leq \beta_k, \tag{42}$$

where

$$M = (A+BK)^TQ(A+BK) + \left(C_y + DK\right)^T\left(C_y+DK\right), \tag{43}$$

and β_k is chosen to be as small as possible by including it in the objective function as follows:

$$J = J_1 + \mathbf{1}_\phi(\mathbf{x}_0,\mathbf{u}_{-1})J_2 + \lambda\sum_{k=0}^{N}\beta_k, \tag{44}$$

with $\lambda>0$ a parameter controlling the trade-off of the *tightness* of the relaxation with respect to the net energy regenerated. The bigger the value of λ, the smaller the resulting net energy regenerated. This is because the constraints become *tighter* and the resulting control expends more energy enforcing them.

Next, we show that the feasibility of the LMI in Eq. (33) is sufficient to guarantee the feasibility of the constraints $\mathbf{C}_1$ and $\mathbf{C}_2$ with $\beta_k=0$. Thus, for sufficiently large λ, the solution of the relaxation will converge to a feasible point of the original problem.

Theorem 4.1. *If the LMI in Eq. (33) is feasible, then*

$$\mathbf{x}_k^TM\mathbf{x}_k + \|\mathbf{x}_0\|_P^2 - 2\mathbf{x}_0^TP\mathbf{x}_k \leq 0, \tag{45}$$

for all $\mathbf{x}_0\in\mathbb{R}^n$.

Proof. The LMI in Eq. (33) is equivalent to

$$M - P \preceq 0. \tag{46}$$

From Eq. (45), it follows that

$$\| \mathbf{x}_k - M^{-1} P \mathbf{x}_0 \|_M^2 - \mathbf{x}_0^T \left(P M^{-1} P - P \right) \mathbf{x}_0 \le 0, \tag{47}$$

from which it is clear that Eq. (45) is satisfied if and only if

$$P M^{-1} P - P \succeq 0. \tag{48}$$

Moreover, it follows from $\left(M^{-1/2} P - M^{1/2} \right)^T \left(M^{-1/2} P - M^{1/2} \right) \succeq 0$ that

$$P M^{-1} P - P \succeq P - M. \tag{49}$$

Therefore Eq. (45) is a sufficient condition for Eq. (48), thus completing the proof. □

Remark. It is straightforward to see that the feasibility of $\mathbf{C}_2$ is sufficient for the feasibility of $\mathbf{C}_1$, since there exists at least one $\mathbf{u}_k$ $(=K\mathbf{x}_k)$ such that $\mathbf{C}_1$ is feasible.

Remark. Furthermore, heuristic extensions of $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, are obtained by neglecting the affine terms and requiring that the quadratic terms be small as possible with respect to the weighted objective:

$$\mathbf{C}_3 : \| A\mathbf{x}_k + B\mathbf{u}_k \|_Q^2 + \| \mathbf{y}_k \|^2 \le \beta_k, \tag{50}$$

$$\mathbf{C}_4 : \mathbf{x}_k^T M \mathbf{x}_k \le \beta_k. \tag{51}$$

Remark. The slack variable β_k included in $\mathbf{C}_i$, $i = 1, 2, 3, 4$, can help with possible numerical instability.

Remark. The relaxed constraints $\mathbf{C}_i$, $i = 1, 2, 3, 4$, restrict the states of the system to live within a bounded region whose determined by β_k, which is in turn required to be as small as possible in the resulting optimization problem. Thus, provided that the feasibility of the constraints is guaranteed, the resulting closed-loop system is provably stable.

Finally, the relaxed MPC problem is expressed as

$$\text{minimize} \quad J = J_1 + \mathbf{1}_\phi(\mathbf{x}_0, \mathbf{u}_{-1}) J_2 + \lambda \sum_{k=0}^{N} \beta_k \tag{52}$$

$$\text{subject to} : \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \quad \mathbf{x}_0 = \mathbf{x}(t), \tag{53}$$

$$\| \mathbf{u}_k \| \le \overline{u}, \tag{54}$$

$$\text{either } \mathbf{C}_1 \text{ or } \mathbf{C}_2 \text{ or } \mathbf{C}_3 \text{ or } \mathbf{C}_4 \tag{55}$$

5. Output-feedback consideration

Here, the formulations in the previous section are extended to the cases where not all the states are directly measurable with the aid of sensors but need to be estimated. The system model is then given in a more general form as

$$\mathbf{G} : \begin{cases} \mathbf{x}(i+1) = A\mathbf{x}(i) + B\mathbf{u}(i) + B_w \mathbf{w}(i) \\ \mathbf{y}(i) = C_y \mathbf{x}(i) + D\mathbf{u}(i) \\ \mathbf{y}_m(i) = C_m \mathbf{x}(i), \end{cases} \tag{56}$$

where $\mathbf{y}_m(i)$ denotes the vector of measured signals. Let $\hat{\mathbf{x}}(i)$ be the output of an observer designed to estimate the state $\mathbf{x}(i)$ from the measurement $\mathbf{y}_m(i)$. While the design of a robust observer is not a goal of this paper, it is however required that the state estimation error satisfies the following:

$$\mathbf{e}(i) \triangleq \mathbf{x}(i) - \hat{\mathbf{x}}(i) \in \mathbb{B}_r(0), \tag{57}$$

for some $r \in \mathbb{R}_+$. The topic of the design of such observer is well studied in the literature [36–39]. Also, let

$$\mathcal{P} = \mathcal{C}\mathrm{o}\left(\hat{\mathbf{x}}_{0_1}, \hat{\mathbf{x}}_{0_2}, \ldots, \hat{\mathbf{x}}_{0_n}\right) \tag{58}$$

be the smallest invariant polytope of the estimation error. See references [40,41] and references therein on how to compute such polytope. Here, $\mathcal{C}\mathrm{o}(V_1, V_2, \ldots V_L)$ refers to the convex hull whose vertices are given by V_j, $j = 1, 2, \ldots L$. Thus, the state of the system at time $t = iT_s$ can be expressed as the following convex combination:

$$\mathbf{x}(i) = \sum_{j=1}^{n} \theta_j \left(\hat{\mathbf{x}}(i) + \hat{\mathbf{x}}_{0_j}\right) \triangleq \sum_{j=1}^{n} \theta_j \hat{\mathbf{x}}_j(i) \tag{59}$$

for some $\theta_j \in \mathbb{R}_+$ satisfying

$$\sum_{j=1}^{n} \theta_j = 1. \tag{60}$$

Let

$$\mathbf{X} = \left[\mathbf{x}_0^T, \ \mathbf{x}_1^T, \ \ldots, \ \mathbf{x}_{N+1}^T\right]^T, \tag{61}$$

$$\mathbf{U} = \left[\mathbf{u}_0^T, \ \mathbf{u}_1^T, \ \ldots, \ \mathbf{u}_N^T\right]^T, \tag{62}$$

$$\boldsymbol{\beta} = \left[\beta_0, \ \beta_1, \ \ldots, \ \beta_N\right]^T, \tag{63}$$

and the convex set $\Pi(\chi)$ defined as

$$\Pi(\chi) = \left\{ (\mathbf{X}, \mathbf{U}, \boldsymbol{\beta}) : \begin{pmatrix} \mathbf{x}_0 = \chi, \\ \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \\ \|\mathbf{u}_k\| \le \overline{u}, \\ \text{either } \mathbf{C}_1 \text{ or } \mathbf{C}_2 \text{ or } \mathbf{C}_3 \text{ or } \mathbf{C}_4 \end{pmatrix}_{k=0}^{N} \right\}. \tag{64}$$

Then, the relaxed MPC problem can be expressed as

$$\text{minimize} \quad J(\mathbf{X}, \mathbf{U}, \boldsymbol{\beta}) = J_1 + \mathbf{1}_\phi(\mathbf{x}_0, \mathbf{u}_{-1}) J_2 + \lambda \sum_{k=0}^{N} \beta_k \tag{65}$$

$$\text{subject to}: \quad (\mathbf{X}, \mathbf{U}, \boldsymbol{\beta}) \in \Pi(\mathbf{x}(t)). \tag{66}$$

Lemma 5.1. *If* $\|A\mathbf{x}_j + B\mathbf{u}\|_Q^2 \le \beta, \ \forall j = 1, \ldots n$ *and some* $Q \in \mathbb{S}_+$, *then* $\|A \sum_{j=1}^{n} \theta_j \mathbf{x}_j + B\mathbf{u}\|_Q^2 \le \beta$ *for all* $\boldsymbol{\theta} \in \mathbb{R}_+^n \triangleq [\theta_1, \theta_2, \ldots, \theta_n]]^T$, $\sum_{j=1}^{n} \theta_j = 1$.

Proof. It is straightforward to verify that

$$A\sum_{j=1}^{n}\theta_i\mathbf{x}_j+B\mathbf{u}=\sum_{j=1}^{n}\theta_i\left(A\mathbf{x}_j+B\mathbf{u}\right). \tag{67}$$

Thus, using the Triangle Inequality, it follows that

$$\left\|A\sum_{j=1}^{n}\theta_j\mathbf{x}_j+B\mathbf{u}\right\|_Q^2\le\sum_{j=1}^{n}\theta_j\|A\mathbf{x}_j+B\mathbf{u}\|_Q^2 \tag{68}$$

$$\left\|A\sum_{j=1}^{n}\theta_j\mathbf{x}_j+B\mathbf{u}\right\|_Q^2\le\sum_{j=1}^{n}\theta_j\beta=\beta. \quad \square \tag{69}$$

Corollary 5.2. *If* $\sum_{j=1}^{n}\|A\mathbf{x}_j+B\mathbf{u}\|_Q^2\le\beta$, *for some* $Q\in\mathbb{S}_+$, *then* $\|A\sum_{j=1}^{n}\theta_j\mathbf{x}_j+B\mathbf{u}\|_Q^2\le\beta$ *for all* $\boldsymbol{\theta}\in\mathbb{R}_+^n\triangleq[\theta_1,\theta_2,\ldots,\theta_n]^T$, $\sum_{j=1}^{n}\theta_j=1$.

Proof. The results follow from the result of Lemma 5.1 by noting that $\sum_{j=1}^{n}\|A\mathbf{x}_j+B\mathbf{u}\|_Q^2\le\beta$ implies that $\|A\mathbf{x}_j+B\mathbf{u}\|_Q^2\le\beta,\ \forall j=1,\ldots n.$ □

The following theorem demonstrates how robustness to state estimation error can be achieved by enforcing the constraints in Eq. (66) at the vertices of the observer invariant-polytope.

Theorem 5.3. *Given a sequence* $\boldsymbol{x}_j,\ j=1,\ldots,n$, *if there exist corresponding* $\boldsymbol{X}_j$, *and* $\boldsymbol{U}$ *such that*

$$\left(\boldsymbol{X}_j,\boldsymbol{U},\boldsymbol{\beta}\right)\in\Pi\left(\boldsymbol{x}_j\right),\quad j=1,\ldots,n, \tag{70}$$

then, there exists $\boldsymbol{X}(\boldsymbol{\theta})$ *such that*

$$(\boldsymbol{X}(\boldsymbol{\theta}),\boldsymbol{U},\boldsymbol{\beta})\in\Pi\left(\sum_{j=1}^{n}\theta_j\boldsymbol{x}_j\right) \tag{71}$$

for all $\boldsymbol{\theta}\in\mathbb{R}_+^n\triangleq[\theta_1,\theta_2,\ldots,\theta_n]^T$ *satisfying*

$$\sum_{j=1}^{n}\theta_j=1. \tag{72}$$

Proof. Using the prediction model, it straightforward to verify that

$$\mathbf{X}(\boldsymbol{\theta})=\left[\left(\sum_{j=1}^{n}\theta_j\mathbf{x}_j\right)^T\ \left(A\sum_{j=1}^{n}\theta_j\mathbf{x}_j+B\mathbf{u}_0\right)^T\ \cdots\right.$$
$$\left.\left(A^{N+1}\sum_{j=1}^{n}\theta_j\mathbf{x}_j+\sum_{j=0}^{N}A^{N-j}B\mathbf{u}_j\right)^T\right]^T. \tag{73}$$

Thus, if Eq. (70) holds, using Lemma 5.1, it follows that Eq. (71) holds. To see this, it is sufficient to show that the constraints $\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3$ and $\mathbf{C}_4$ are satisfied for $(\mathbf{x}_k(\boldsymbol{\theta}),\mathbf{U},\boldsymbol{\beta})$, where

$$\mathbf{x}_k(\boldsymbol{\theta})=A^k\sum_{j=1}^{n}\theta_j\mathbf{x}_j+\sum_{j=0}^{k-1}A^{k-1-j}B\mathbf{u}_j. \tag{74}$$

The conclusion follows by noting that

$$\| A\mathbf{x}_k(\boldsymbol{\theta}) + B\mathbf{u}_k \|_Q^2 = \left\| A^{k+1}\sum_{j=1}^{n}\theta_j\mathbf{x}_j + \sum_{j=0}^{k}A^{k-j}B\mathbf{u}_j\right\|_Q^2 \tag{75}$$

$$\| \mathbf{y}_k(\boldsymbol{\theta}) \|^2 = \left\| C_y A^k\sum_{j=1}^{n}\theta_j\mathbf{x}_j + \sum_{j=0}^{k-1}A^{k-1-j}B\mathbf{u}_j + D\mathbf{u}_k\right\|^2 \tag{76}$$

$$\| \mathbf{x}_K(\boldsymbol{\theta}) \|_M^2 = \left\| A^k\sum_{j=1}^{n}\theta_j\mathbf{x}_j + \sum_{j=0}^{k-1}A^{k-1-j}B\mathbf{u}_j\right\|_M^2 \tag{77}$$

and using Lemma 5.1. □

Remark. Theorem 5.3 shows that by requiring the solution of the relaxed MPC to be feasible at the vertices of the polytope $\mathcal{P}$ translated by observed state $\hat{\mathbf{x}}(t)$, the resulting closed-loop system is robustly feasible with respect to the state estimation error. To this effect, the output-feedback relaxed MPC problem is given by

$$\mathbf{MPC}_1\begin{cases} \text{minimize} & J\left(\dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\mathbf{X}_j, \mathbf{U}, \boldsymbol{\beta}\right) \\ \text{subject to :} & \left(\mathbf{X}_j, \mathbf{U}, \boldsymbol{\beta}\right) \in \Pi(\hat{\mathbf{x}}(t) + \hat{\mathbf{x}}_{0_j}), \quad j = 1, \dots, n. \end{cases} \tag{78}$$

While the MPC problem $\mathbf{MPC}_1$ allows for robustness with respect to the state estimation error, the number of constraints was increased by a factor of n. Fortunately, using Corollary 5.2, the total number of constraint to ensure robustness to state estimation error can be reduced to $nN-1$. The following theorem demonstrates how robustness to state estimation error can be achieved by using a single dissipativity constraint. First, we introduce new constraints based on $\mathbf{C}_i$, $i = 1, 2, 3, 4$, as follows:

$$\sum \mathbf{C}_1 : \sum_{j=1}^{n}\left(\| A\mathbf{x}_{k,j} + B\mathbf{u}_k \|_Q^2 + \| \mathbf{y}_{k,j} \|^2 + \| \mathbf{x}_0 \|_P^2 - 2\mathbf{x}_0^T P\mathbf{x}_{k,j}\right) \le \beta_k, \tag{79}$$

$$\sum \mathbf{C}_2 : \sum_{j=1}^{n}\left(\mathbf{x}_{k,j}^T M\mathbf{x}_{k,j} + \| \mathbf{x}_0 \|_P^2 - 2\mathbf{x}_0^T P\mathbf{x}_{k,j}\right) \le \beta_k, \tag{80}$$

$$\sum \mathbf{C}_3 : \sum_{j=1}^{n}\left(\| A\mathbf{x}_{k,j} + B\mathbf{u}_k \|_Q^2 + \| \mathbf{y}_{k,j} \|^2\right) \le \beta_k, \tag{81}$$

$$\sum \mathbf{C}_4 : \sum_{j=1}^{n}\left(\mathbf{x}_{k,j}^T M\mathbf{x}_{k,j}\right) \le \beta_k, \tag{82}$$

where $\mathbf{x}_{k,j}$ is the kth step prediction starting from the initial state $\hat{\mathbf{x}}(t) + \hat{\mathbf{x}}_{0_j}$, and $\mathbf{y}_{k,j} = C_y\mathbf{x}_{k,j} + D\mathbf{u}_k$. Note that the control is not indexed by j. This is because a unique control is required, as in Theorem 5.3, to ensure feasibility over the polytope $\mathcal{P}$. Consequently, the convex set $\sum \Pi(\chi)$ is defined as

$$\sum \Pi(\chi) = \left\{ (\mathbf{X}_1, \ldots, \mathbf{X}_n, \mathbf{U}, \boldsymbol{\beta}) : \begin{pmatrix} \mathbf{x}_0 = \chi + \hat{\mathbf{x}}_{0_j}, \; j = 1, \ldots n, \\ \mathbf{x}_{k+1,j} = A\mathbf{x}_{k,j} + B\mathbf{u}_k, \; j = 1, \ldots n, \\ \| \mathbf{u}_k \| \le \overline{u}, \\ \text{either } \sum \mathbf{C}_1 \text{ or } \sum \mathbf{C}_2 \text{ or } \sum \mathbf{C}_3 \text{ or } \sum \mathbf{C}_4 \end{pmatrix}_{k=0}^{N} \right\}, \tag{83}$$

where

$$\mathbf{X}_j = \left[\mathbf{x}_{0,j}^T, \;\; \mathbf{x}_{1,j}^T, \;\; \ldots, \;\; \mathbf{x}_{N+1,j}^T \right]^T \tag{84}$$

$$\mathbf{U} = \left[\mathbf{u}_0^T, \;\; \mathbf{u}_1^T, \;\; \ldots, \;\; \mathbf{u}_N^T \right]^T, \tag{85}$$

$$\boldsymbol{\beta} = \left[\beta_0, \;\; \beta_1, \;\; \ldots, \;\; \beta_N \right]^T. \tag{86}$$

Theorem 5.4. *If $\sum \Pi(\chi)$ is feasible for some $\chi \in \mathbb{R}^n$, then $\Pi(\chi + \sum_{j=1}^{n} \theta_j \hat{\mathbf{x}}_{0_j})$ is feasible for all $\boldsymbol{\theta} \in \mathbb{R}_+^n \triangleq [\theta_1, \theta_2, \ldots, \theta_n]^T$ satisfying*

$$\sum_{j=1}^{n} \theta_j = 1. \tag{87}$$

Proof. Suppose $\sum \Pi(\chi)$ is feasible, and that $(\mathbf{X}_1, \ldots, \mathbf{X}_n, \mathbf{U}, \boldsymbol{\beta}) \in \sum \Pi(\chi)$ for some $\chi \in \mathbb{R}^n$. Given any $\boldsymbol{\theta}$ satisfying Eq. (87), let

$$\mathbf{x}_k(\boldsymbol{\theta}) = A^k \sum_{j=1}^{n} \theta_j \mathbf{x}_{0,j} + \sum_{j=0}^{k-1} A^{k-1-j} B\mathbf{u}_j. \tag{88}$$

It follows that

$$\begin{aligned} \mathbf{x}_{k+1}(\boldsymbol{\theta}) &= A^{k+1} \sum_{j=1}^{n} \theta_j \mathbf{x}_{0,j} + \sum_{j=0}^{k} A^{k-j} B\mathbf{u}_j \\ &= A\left(A^k \sum_{j=1}^{n} \theta_j \mathbf{x}_{0,j} + \sum_{j=0}^{k-1} A^{k-1-j} B\mathbf{u}_j \right) + B\mathbf{u}_k \\ &= A\mathbf{x}_k(\boldsymbol{\theta}) + B\mathbf{u}_k. \end{aligned} \tag{89}$$

Moreover, it is clear that $\| \mathbf{u}_k \| \le \overline{u}, \; \forall k = 0, \ldots N$. Also, using Corollary 5.2, it follows that the feasibility of $\sum \mathbf{C}_i$ implies the feasibility of $\mathbf{C}_i$, $i = 1, 2, 3, 4$. Therefore, $(\mathbf{x}_k(\boldsymbol{\theta}), \mathbf{U}, \boldsymbol{\beta}) \in \Pi(\chi + \sum_{j=1}^{n} \theta_j \hat{\mathbf{x}}_{0_j})$. □

Remark. In the light of Theorem 5.4, the output-feedback relaxed MPC problem is given by

$$\mathbf{MPC}_2 \begin{cases} \text{minimize} & J\left(\frac{1}{n} \sum_{j=1}^{n} \mathbf{X}_j, \mathbf{U}, \boldsymbol{\beta} \right) \\ \text{subject to :} & \left(\mathbf{X}_j, \mathbf{U}, \boldsymbol{\beta} \right) \in \sum \Pi(\hat{\mathbf{x}}(t)), \quad j = 1, \ldots, n. \end{cases} \tag{90}$$

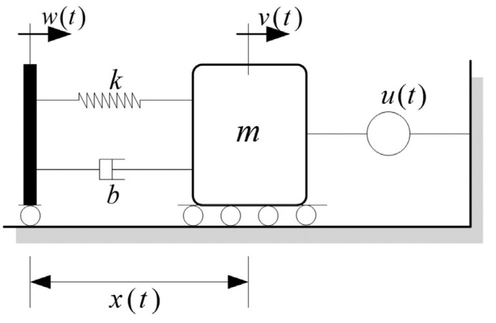

Fig. 1. Single degree-of-freedom vibrating system.

6. Numerical simulation example

In order to validate and compare the approaches presented in the previous sections, a problem of maximizing energy regeneration from a single degree-of-freedom system, subject to a level-set constraint on some performance metric, is considered. The system considered is shown schematically in Fig. 1 and the equations of motion are given by

$$\begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{b}{m} & \frac{k}{m} \\ -1 & 0 \end{bmatrix}}_{A_m} \begin{bmatrix} v \\ x \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix}}_{B_m} u + \underbrace{\begin{bmatrix} \frac{b}{m} \\ 1 \end{bmatrix}}_{B_{m_w}} w, \tag{91}$$

where $w(t)$ and $u(t)$ are the exogenous and controlled input respectively. The measured output is given as

$$y_{\text{meas}} = \underbrace{[1 \;\; 0]}_{C_m} \begin{bmatrix} v \\ x \end{bmatrix}. \tag{92}$$

The system is discretized using a zero-order-hold for a sampling period of T_s. As a result, the dynamics in Eq. (9), together with the objective in Eq. (1), is obtained as

$$A = e^{A_m T_s}, \quad B = \int_0^{T_s} e^{A_m \tau}\, d\tau B_m, \quad B_w = \int_0^{T_s} e^{A_m \tau}\, d\tau B_{m_w}$$

$$C_y = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = [1 \;\; 0].$$

Thus, the objective of the design is to maximize the energy regeneration from the vibrating system using the control input $u(t)$ while minimizing a weighted vector of the velocity $v(t)$ and the relative displacement $x(t)$ of the vibrating mass m with respect to the vibrating source with an exogenous input $w(t)$. It is clear that these are competing objectives. As a result, using the framework developed in the previous sections, the energy regeneration objective is targeted subjected to a performance constraint given by a level-set constraint on the $\mathcal{L}_2-$gain of a weighted vector of the velocity $v(t)$ and the relative displacement $x(t)$ with respect to the exogenous input $w(t)$. The performance level is governed by the value of γ introduced in Section 3.

Table 1
Simulation parameter values.

Parameter	Value
m	100 kg
k	10 000 N/m
b	10 N s/m
T_s	0.01 s
W_1	1
W_2	1
γ	30

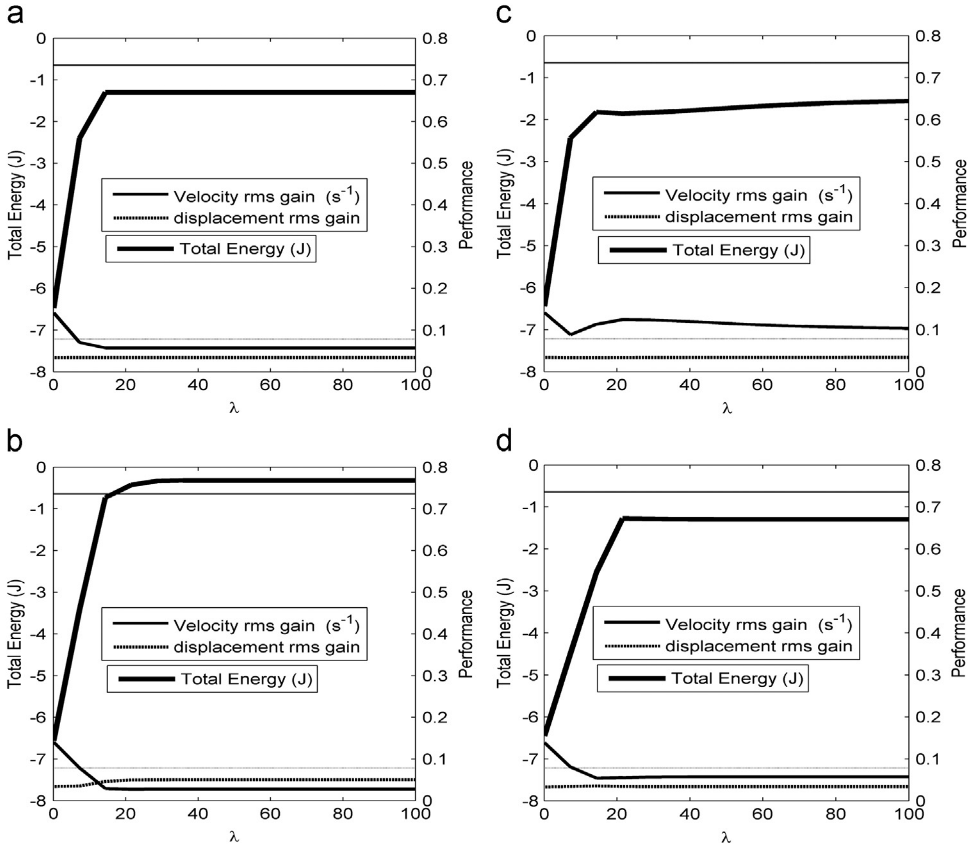

Fig. 2. Simulation results: $\mathbf{MPC}_1$ (the horizontal lines are that rms gains of the passive system: the thicker one corresponds to the velocity and the thinner one to the displacement). (a) Convexification $\mathbf{C}_1$. (b) Convexification $\mathbf{C}_2$. (c) Convexification $\mathbf{C}_3$. (d) Convexification $\mathbf{C}_4$.

Since only the state $v(t)$ is assumed to be measured, the other state of the system is estimated using the Luenberger observer

$$\begin{bmatrix} \hat{v}_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = A \begin{bmatrix} \hat{v}_k \\ \hat{x}_k \end{bmatrix} + Bu_k + L\left(y_{\text{meas}} - C_m \begin{bmatrix} \hat{v}_k \\ \hat{x}_k \end{bmatrix} \right), \tag{93}$$

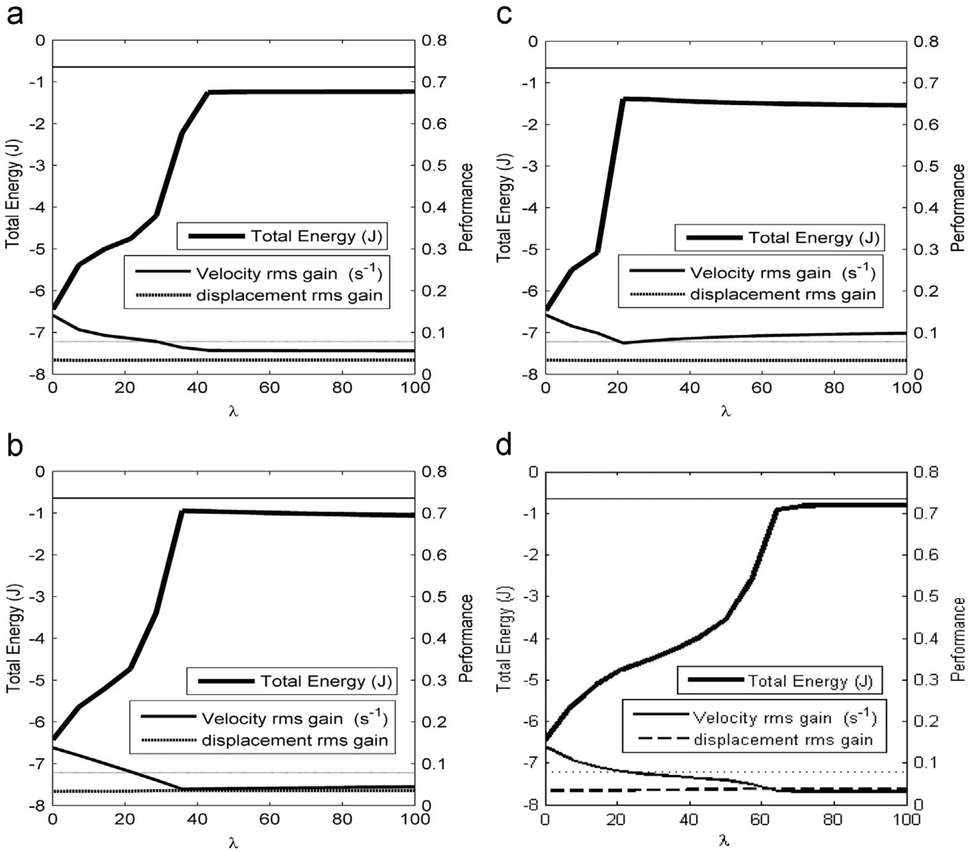

Fig. 3. Simulation results: $\mathbf{MPC}_2$ (the horizontal lines are that rms gains of the passive system: the thicker one corresponds to the velocity and the thinner one to the displacement). (a) Convexification $\sum \mathbf{C}_1$. (b) Convexification $\sum \mathbf{C}_2$. (c) Convexification $\sum \mathbf{C}_3$. (d) Convexification $\sum \mathbf{C}_4$.

where the observer gain L is designed by minimizing the $\mathcal{L}_2-$gain of the estimation error $([v\ x]-[\hat{v}\ \hat{x}])^T$ from the exogenous disturbance $w(t)$. This is achieved by solving the LMI

$$\text{minimize} \quad \eta \tag{94}$$

$$\text{subject to}: \quad \begin{bmatrix} -P+I & 0 & (PA-QC)^T \\ 0 & -\eta & B_w^T P \\ (PA-QC) & PB_w & -P \end{bmatrix} \preceq 0 \tag{95}$$

for the matrix variables P and Q, and setting $L=P^{-1}Q$.

Next, the output-feedback relaxed MPC problems $\mathbf{MPC}_1$ and $\mathbf{MPC}_2$ are solved using the state estimation error invariant polytope

$$\mathcal{P}=\mathcal{C}\text{o}\left(r\begin{bmatrix}-1\\1\end{bmatrix}, r\begin{bmatrix}1\\1\end{bmatrix}, r\begin{bmatrix}-1\\-1\end{bmatrix}, r\begin{bmatrix}1\\-1\end{bmatrix}\right), \quad r\in\mathbb{R}_+.$$

The numerical values of the parameters used in the simulation study are given in Table 1.

The simulation study was carried out for different values of the objective weighting parameter λ. For each value of λ, the closed loop system is simulated for 10 s using a uniform random disturbance with the interval $[-0.005, 0.005]$ m. The *rms gain*[5] of the velocity $v(t)$ and the relative displacement $x(t)$ of the closed loop and the passive system,[6] as well as the total energy are plotted against λ. Figs. 2 and 3 show the results for the different convexification approaches for $\mathbf{MPC}_1$ and $\mathbf{MPC}_2$ respectively. The rms gains of the corresponding passive systems are plotted using smaller line widths. It is seen that the closed loop system facilitated energy dissipation and outperformed[7] the passive system in all cases. As expected, it is also seen that $\mathbf{MPC}_2$ is a little less conservative than $\mathbf{MPC}_1$, thereby regenerating more energy.

7. Conclusions

The problem of optimal energy-regenerative model predictive control for linear systems subject to stability/dissipativity constraints, as well as some hard constraints on the state and control vectors was considered. Some convex relaxation approaches of the original nonconvex MPC problem were considered. As a result, a solution to the original problem was proposed by solving a more relaxed convex MPC problem. The feasibility and stability of the resulting MPC problem were also examined. Moreover, output feedback considerations for the formulated MPC problem were presented. Simulation results using a single degree of freedom vibrating system were used to validate the theoretical claims made in the paper.

While robustness to exogenous disturbance was properly accounted for in this paper, robustness to model uncertainties was not considered. In future, efforts will be directed to extend the methods presented to include robustness to structured and unstructured uncertainties. Moreover, the methods will also be extended to nonlinear model predictive control (NMPC) problems.

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[5]The rms gain is defined as the ratio of the rms value of the signal in question and the rms of the disturbance profile.
[6]Without control.
[7]With respect to the considered objectives.

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